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Optimal Approximation and Characterization of the Error and Stability Functions in Banach Spaces*

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INTRODUCTION

In several problems of approximation theory we have to use the error function and the stability function of a space P of approximants. (See, e.g. Refs. [2] and [7]). Namely, suppose U and V are Banach spaces such that

$$U \subset V \text{ and the injection is compact and dense.} \tag{1}$$

Let P be a closed subspace of V. We define the error function by

$$e_U{}^V(P) = \sup_{u \in U} \inf_{v \in P} \frac{\| u - v \|_V}{\| u \|_U} . \tag{2}$$

If P is finite-dimensional, we define the stability function by

$$s_U{}^V(P) = \sup_{u \in P} \frac{\| u \|_U}{\| u \|_V} . \tag{3}$$

($S_U{}^V(P)$ is finite, since all norms on a finite-dimensional space are equivalent). The motiviation for the present paper was the study of these functions.

First of all, they are related by the following duality relation:

$$e_U{}^V(P) = s_{U'}^{V'}(P^\perp); \qquad s_U{}^V(P) = e_{U'}^{V'}(P^\perp), \tag{4}$$

where U', V' are the duals of U and V, respectively, and $P^\perp$ is the annihilator of P. On the other hand, they are eigenvalues of certain nonlinear operators (which are linear in case U and V are Hilbert spaces). For the sake of simpli-

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city, we shall restrict our study to the case where U and V are smooth, uniformly convex, reflexive Banach spaces. In this case, there exists a unique duality mapping J from V onto V' (resp., K from U onto U') which is the one-to-one nonlinear mapping J defined by

$$(Ju, u) = \| u \|_V^2 ; \qquad \| Ju \|_{V'} = \| u \|_V . \tag{5}$$

Then if t is the (nonlinear) best approximation projector from V onto P, defined by

$$\| u - tu \|_V = \inf_{v \in P} \| u - v \|_V , \tag{6}$$

the error function is the square root of the largest eigenvalue of the operator $(1 - t) K^{-1} J(1 - t)$.

In order to characterize the stability function, we have to introduce another nonlinear projector s from V onto P (the stabilization projector). If we define the cosine of the angle between two elements, u and v, of V by

$$\cos(u, v) = \frac{(Ju, v)}{\| u \| \| v \|} , \tag{7}$$

then the projector s is defined by

$$\cos(u, su) = \sup_{v \in P} \cos(u, v); \qquad \| su \| = \cos(u, su) \| u \|. \tag{8}$$

These projectors s and t are linked by the following duality relation:

$$s = J^{-1}(1 - t^{\perp})J, \tag{9}$$

where $t^{\perp}$ is the best approximation projector from V' onto $P^{\perp}$.

When V is a Hilbert space, formula (9) shows that s is the (Hilbertian) adjoint of t; s coincides with t, since the orthogonal projectors are the ones which are self-adjoint.

We shall prove, in general, that if P is a finite-dimensional subspace of U, the stability function is the square root of the largest eigenvalue of the operator $sJ^{-1}Ks$. Incidentally, we shall prove the following two formulas. If $P = \ker r$ is the kernel of a continuous linear operator from V onto a Banach space E, then

$$t = 1 - pr, \qquad \text{where} \quad p = J^{-1}r'(rJ^{-1}r')^{-1},$$

and if $P = pE$ is the closed range of a linear isomorphism from a Banach space E into E, then

$$s = pr, \qquad \text{where} \quad r = (p'Jp)^{-1} p'J.$$

1. Duality Mapping and Cosines

1.1. *Duality mapping*

Let us recall the definition of the *duality mapping* from a Banach space V into its dual V' (see Refs. [3, 4]).

By the Hahn–Banach theorem, we can associate with any u of the unit sphere of V a continuous linear form Ju of the unit sphere of its dual V' such that

$$(Ju, u) = 1; \qquad \| Ju \|_* = 1, \tag{1.1}$$

and we shall choose Ju so that

$$J(-u) = -J(u). \tag{1.2}$$

Here (f, v) denotes the duality pairing on $V' \times V$, $\| \ \|$ is the norm of V, and $\| \ \|_*$ is its dual norm.

We shall extend this mapping J, defined on the unit sphere, to all of V. Let $\alpha > 1$. We set

$$Ju = J_V{}^\alpha u = \| u \|^{\alpha-1} J\left(\frac{u}{\| u \|}\right); \qquad J(0) = 0. \tag{1.3}$$

Such an operator is called an (α-) *duality mapping* from V into V' and satisfies

$$\begin{aligned}
&\text{(i)} && \| Ju \|_* = \| u \|^{\alpha-1};\\
&\text{(ii)} && (Ju, u) = \| u \|^{\alpha};\\
&\text{(iii)} && (Ju - Jv, u - v) \geqslant (\| u \| - \| v \|)(\| u \|^{\alpha-1} - \| v \|^{\alpha-1}) \geqslant 0;\\
&\text{(iv)} && J(\lambda u) = | \lambda |^{\alpha-2} \lambda Ju.
\end{aligned} \tag{1.4}$$

We are mainly interested in the case where there exists a unique bijective duality mapping from V onto V'. This is the case when V is a *smooth, strictly convex, reflexive Banach* space (briefly, an R.S. space), where

(i) A space V is *smooth* iff each point of its unit sphere possesses a unique supporting hyperplane (equivalently: iff the norm $\| u \|$ is Gâteaux-differentiable at each point of the unit sphere);

(ii) A space V is *strictly convex* (or *rotund*) iff its unit sphere does not contain any line segment.

Let us recall (see Ref. [5]) that a reflexive Banach space is smooth iff its dual is strictly convex. Therefore, we have

Lemma 1.1. *Let V be an R.S. space. Then there exists a unique α-duality* *mapping $J = J_V^\alpha$ which is one-to-one from V onto V', and which is equal to the Gâteaux derivative of the functional* $1/\alpha \| v \|^\alpha$. *Moreover,*

$$(J_V^\alpha)^{-1} = J_{V'}^{\alpha'}, \qquad \textit{where} \quad \frac{1}{\alpha} + \frac{1}{\alpha'} = 1. \tag{1.5}$$

The Lebesgue spaces L^α and the Sobolev spaces $W^{m,\alpha}$ are R.S. spaces for $1 < \alpha < +\infty$. The α-duality mapping of L^α is the map $Ju = | u |^{\alpha-2}u$. A closed subspace and a factor space of an R. S. space are also R.S. spaces. The following lemma provides a tool for constructing duality mappings.

LEMMA 1.2. *Let φ_k be a continuous linear operator from a space V into an R.S. space E_k ($k = 0,..., m$) and let V be a Banach space for the norm*

$$\| v \| = \left(\sum_{k=0}^{m} \| \varphi_k v \|_{E_k}^\alpha \right)^{1/\alpha}, \qquad \alpha > 1. \tag{1.6}$$

Then V is also an R.S. space. If J_k is the α-duality mapping from E_k onto E_k', the α-duality mapping of V is

$$J_V^\alpha = \sum_{k=0}^{m} \varphi_k' J_{E_k}^\alpha \varphi_k , \tag{1.7}$$

where φ_k' denotes the transpose of φ_k.

If V is a Hilbert space with the inner product $((u, v))$, the 2-duality mapping from V onto V' is nothing else than the canonical isomorphism of the Riesz–Fischer theorem defined by

$$(Ju, v) = ((u, v)); \qquad J \in \mathscr{L}(V, V'). \tag{1.8}$$

1.2. *Cosine of two vectors of an R.S. space*

We extend the usual definition of the cosine of the angle between two vectors of a Hilbert space in the following way: If V is an R.S. space we define

$$\cos(u, v) = \cos_V(u, v) = \frac{(J_V^\alpha u, v)}{\| u \|^{\alpha-1} \| v \|} = \frac{(J_V^\beta u, v)}{\| u \|^{\beta-1} \| v \|}; \quad \alpha > 1; \quad \beta > 1. \tag{1.9}$$

It is a *non-symmetric* functional on $V \times V$ satisfying

$$| \cos(u, v)| \leqslant 1; \quad \cos(\lambda u, \mu v) = \cos(u, v) \qquad \text{for} \quad \lambda, \mu \geqslant 0 \tag{1.10}$$

and

$$\cos_V(u, v) = \cos_{V'}(Jv, Ju). \tag{1.11}$$

Remark 1.1. If V is a normed linear space, we can also define the cosine of the angle between two vectors, u and v, of the unit ball in the following way:

$$\cos_V(u, v) = \lim_{\substack{\theta \to 0 \\ \theta > 0}} \frac{(\| u + \theta v \| - \| u \|)}{\theta}. \tag{1.12}$$

1.3. *Bounded homogeneous operators*

We shall deal not only with continuous linear operators, but, more generally, with *bounded homogeneous operators* (briefly, H operators), i.e., operators A satisfying

$$\begin{aligned} &\text{(i)} \quad A(\lambda u) = \lambda A u; \quad \lambda \text{ a scalar}, \\ &\text{(ii)} \quad A \text{ maps bounded sets onto bounded sets.} \end{aligned} \tag{1.13}$$

We can associate with such an operator its norm

$$\| A \| = \sup_{\|u\| \leqslant 1} \| Au \| = \sup_{u \neq 0} \frac{\| Au \|}{\| u \|} \tag{1.14}$$

and its kernel

$$\ker A = \{u \text{ such that } Au = 0\}, \tag{1.15}$$

which is a symmetric cone.

We say that an H operator A is a projector if $A^2 = A$.

2. Best Approximation and Stabilization Projectors

2.1. *Best approximation projectors*

Let P be a closed subspace of a Banach space V and consider the number

$$\inf_{v \in P} \| u - v \|. \tag{2.1}$$

If P is a reflexive subspace of V, this inf is achieved on P at least once, and there exists at most one point of P achieving this minimum if V is strictly convex (see Ref. [6]). Therefore, when V is a strictly convex reflexive Banach space, there exists a unique point $tu = t_P u \in P$ achieving the minimum (2.1) *and we call* $t = t_P$ *the best approximation* projector onto P.

Let us recall the following

Lemma 2.1 *The best approximation projector t is an H projector satisfying*

$$\| 1 - t \| = 1. \tag{2.2}$$

We set

$$P^{\oplus} = \ker t = (1 - t)V. \tag{2.3}$$

We also recall the following (see Ref. [8]):

LEMMA 2.2. *If V is an R.S. space, the best approximant tu is characterized by*

$$tu \in P \quad \text{and} \quad J(u - tu) \in P^{\perp}. \tag{2.4}$$

Therefore, J maps $P^{\oplus}$ onto $P^{\perp}$ and P onto $P^{\perp\oplus}$.

2.2. *Stabilization projectors*

Let us consider the number

$$\lambda = \sup_{v \in P} \cos(u, v) = \sup_{v \in P} \frac{(Ju, v)}{\| u \|^{\alpha-1} \| v \|}; \tag{2.5}$$

we can restrict v to belong to the unit sphere of V. Observe that $\cos(u, v) = 0$ on P iff u belongs to $P^{\oplus}$. Otherwise, the supremum is positive.

If P is a reflexive subspace, this sup is achieved at least once on the unit sphere. On the other hand, the set of points of the unit sphere of P achieving this sup is convex. Indeed, if v_0 and v_1 are such points, set

$$v_\alpha = \frac{(1-\alpha)\, v_0 + \alpha v_1}{\|(1-\alpha)\, v_0 + \alpha v_1 \|}, \qquad 0 \leqslant \alpha \leqslant 1.$$

Then

$$\cos(u, v_\alpha) = \frac{\lambda}{\|(1-\alpha)\, v_0 + \alpha v_1 \|} \leqslant \lambda. \tag{2.6}$$

This implies that $\|(1-\alpha)v_0 + \alpha v_1 \| \geqslant 1$ and that v_α also achieves the sup.

Therefore, as in the best approximation problem, there exists a unique point $s^0(u) = s_P{}^0(u)$ of the unit sphere of P achieving the sup in (2.5), *when V is a strictly convex reflexive Banach space.*

In this case, we set

$$su = s_P{}^\alpha(u) = \begin{cases} 0 & \text{iff} \quad u \in P^{\oplus}, \\ \| u \| \cos(u, s_P{}^0(u))^{\alpha'-1}\, s_P{}^0(u) & \text{otherwise}, \end{cases} \tag{2.7}$$

and we *call $s_P{}^\alpha$ the α-stabilization projector onto P.* Indeed, as one can check, we have

LEMMA 2.3. *The operator $s_P{}^\alpha$, defined by* (2.7), *is an H projector of norm* 1, *satisfying*

$$\| s_P{}^\alpha u \|^{\alpha-1} = \cos(u, s_P{}^\alpha u) \| u \|^{\alpha-1} \leqslant \| u \|^{\alpha-1} \tag{2.8}$$

and

$$\ker s_P^\alpha = (1 - s_P^\alpha)V = P^\oplus. \tag{2.9}$$

When V is a Hilbert space (and $\alpha = 2$) the best approximation operator and the stabilization projector coincide with the orthogonal projector onto P (see the remark following Lemma 1.2). When V is an R.S. space, then, as we shall see, in some sense s_P^α is the "adjoint" of t_P.

THEOREM 2.1. *Let V be an R.S. space, P a closed subspace of V, and $P^\perp$ its annihilator. Let $J = J_V^\alpha$ be the α-duality mapping from V onto V'. Then the stabilization projector $s = s_P^\alpha$ onto P and the best approximation projector $t^\perp = t_{P^\perp}$ onto $P^\perp$, in V', are related by the formula*

$$s = J^{-1}(1 - t^\perp)J. \tag{2.10}$$

Indeed, to maximize $\cos(u, v)$ on P amounts to maximizing, on P, the function

$$\rho(v) = \| u \|^{\alpha(\alpha-1)} \mid \cos(u, v)|^{\alpha-1} \cos(u, v) = \frac{|(Ju, v)|^{\alpha-1} (Ju, v)}{\| v \|^\alpha}, \tag{2.11}$$

where $u \notin P^\oplus$. Since V is smooth, the functional $\rho(v)$ is Gâteaux-differentiable at every point $v \neq 0$ and its derivative at v_0 is

$$Lv_0 = \frac{\| v_0 \|^\alpha \, |(Ju, v_0)|^{\alpha-1} Ju - |(Ju, v_0)|^{\alpha-1} (Ju, v_0) Jv_0}{\| v_0 \|^{2\alpha}}. \tag{2.12}$$

Let $v_0 = s_P^0(u)$ be the point of the unit sphere of P achieving the sup of $\rho(v)$ on P. Since $\rho(v) \leqslant \rho(v_0)$ for any v in P, we deduce that $(Lv_0, v) = 0$ for any such v, and thus, that Lv_0 belongs to $P^\perp$. In other words, there exists an f belonging to $P^\perp$ such that

$$Ju - (Ju, v_0) Jv_0 = f. \tag{2.13}$$

But since $Ju - f = (Ju, v_0) Jv_0$ belongs to $P^{\perp\oplus}$ (by the Lemma 2.2), we deduce that $f = t^\perp Ju = t_{P^\perp} Ju$. Therefore, $(Ju, v_0) Jv_0 = (1 - t^\perp) Ju$ and, since $v_0 = s_P^0(u)$,

$$J^{-1}(1 - t^\perp) Ju = |(Ju, v_0)|^{\alpha'-2} (Ju, v_0) v_0 = s_P^\alpha(u).$$

Conversely, let us assume that s_P^α is defined by

$$Js_P^\alpha(u) = (1 - t^\perp) Ju. \tag{2.14}$$

If u belongs to $P^{\oplus}$, it follows that $s_P{}^{\alpha}(u) = 0$. Otherwise,

$$(Js_P{}^{\alpha}(u) - Ju, v) = -(t^{\perp}Ju, v) = 0 \qquad \text{for any} \quad v \text{ in } P.$$

Taking $v = s_P{}^{\alpha}(u)$, we get

$$\cos(u, s_P{}^{\alpha}(u)) = \frac{(Ju, s_P{}^{\alpha}(u))}{\| u \|^{\alpha-1} \| s_P{}^{\alpha}(u)\|} = \frac{(Js_P{}^{\alpha}(u), s_P{}^{\alpha}(u))}{\| s_P{}^{\alpha}(u)\| \| u \|^{\alpha-1}} = \frac{\| s_P{}^{\alpha}(u)\|^{\alpha-1}}{\| u \|^{\alpha-1}}$$

and (2.8) is satisfied. On the other hand, $s_P{}^{\alpha}(u)$ maximizes $\cos(u, v)$ on P, since

$$| \cos(u, v)| = \frac{|(Ju, v)|}{\| u \|^{\alpha-1} \| v \|} = \frac{|(Js_P{}^{\alpha}(u), v)|}{\| u \|^{\alpha-1} \| v \|} \leqslant \frac{\| s_P{}^{\alpha}(u)\|^{\alpha-1}}{\| u \|^{\alpha-1}} = \cos(u, s_P{}^{\alpha}u).$$

Therefore, $s_P{}^{\alpha}(u)$, defined by (2.14), satisfies (2.7).

Remark 2.1. If V is a Hilbert space, $1 - t^{\perp}$ is equal to the transpose t' of the orthogonal projector t onto P, and $J^{-1}t'J$ is the adjoint of the operator t. Since t is self-adjoint, $s = t$.

2.3. *Characterization of the best approximation projectors*

Suppose a closed subspace R is the kernel of a continuous linear operator r, mapping V onto a Banach space E,

$$R = \ker r; \qquad r \in \mathscr{L}(V, E); \qquad r(V) = E; \tag{2.15}$$

we shall prove a formula expressing the best approximation projector $t = t_R$ on R in terms of r and the duality mapping J from V onto V'.

THEOREM 2.2. *Assume* (2.15) *and that V is an R.S. space. Then the best approximation projector $t = t_R$ onto R satisfies*

$$t = (1 - pr), \tag{2.16}$$

where p is the H operator from E onto $R^{\oplus}$, defined by

$$p = J^{-1}r'(rJ^{-1}r')^{-1}. \tag{2.17}$$

The proof is quite obvious. We have, first of all, to verify that $rJ^{-1}r'$ is invertible. But this is a consequence of Lemma 1.2, which implies that $rJ^{-1}r'$ is the duality mapping from E' onto E, when E' is equipped with the norm $\| e' \|_{E'} = \| r'e' \|_{V'}$, and E, with its dual norm. By a theorem of Banach, E is an R.S. space. Therefore, by Lemma 1.1, $rJ^{-1}r'$ is invertible, and the formula (2.17) is meaningful. Thus, tu belongs to R since

$$rtu = ru - rpru = 0,$$

and tu is the best approximant of u since, by Lemma 2.2, $J(u - tu) = J(pru) = r'(rJ^{-1}r')^{-1}$ belongs to $r'E' = R^{\perp}$.

Incidentally, we solve the problem of "optimal interpolation" in Banach spaces (see, e.g., Ref. [1]) which amounts to finding a right inverse of r having "minimal norm."

For this purpose, let us associate with r and the norm of V the following norm on E:

$$\| e \|_E = \sup_{e' \in E'} \frac{|(e', e)|}{\| r'e' \|_{V'}}. \tag{2.18}$$

For this norm, r is an operator of norm 1. Therefore:

Corollary 2.1. *Let r be a given operator from an R.S. space V onto a Banach space E, equipped with the norm* (2.18). *Then*

$$p = J^{-1}r'(rJ^{-1}r')^{-1} \tag{2.19}$$

is a right inverse of r,

$$rpe = e \qquad \text{for any} \quad e \in E, \tag{2.20}$$

p is an H-isometry,

$$\| pe \|_V = \| e \|_E \qquad \text{for any} \quad e \in E, \tag{2.21}$$

and

$$\| pe \|_V \leqslant \| u \|_V \qquad \text{for any } u \text{ such that } ru = e. \tag{2.22}$$

Indeed, $\| e \|_E$, defined by (2.18), is nothing else than the dual norm of $\| e' \|_E = \| r'e' \|_{V'}$. Therefore, we can set

$$J_E = J_E{}^{\alpha} = (J_{E'}^{\alpha'})^{-1} = (rJ^{-1}r')^{-1}.$$

Then (2.21) follows from

$$\| pe \|^{\alpha} = (Jpe, pe) = (r'J_E{}^{\alpha}e, pe) = (J_E{}^{\alpha}e, e) = \| e \|_E^{\alpha},$$

and (2.22) follows from

$$\begin{aligned} \| pe \|^{\alpha} &= (J_E{}^{\alpha}e, e) = (J_E{}^{\alpha}e, ru) = (r'J_E{}^{\alpha}e, u) \\ &= (Jpe, u) \leqslant \| pe \|^{\alpha-1} \| u \|. \end{aligned}$$

Let us extend this result to general normed linear spaces. Assume that

(i) E is reflexive;

(ii) There exists a duality mapping $L = L^{\alpha}{}_{v'}$ from V' into V'' (2.23) such that $Lr'E' \subset V$.

By Lemma 1.2, $L_E = rLr'$ is a duality mapping from E' *onto* E (by Ref. [4]). Among the duality mappings from E onto E', let us denote by J_E the one which satisfies

$$L_E J_E e = e \qquad \text{for any} \quad e \in E. \tag{2.24}$$

COROLLARY 2.2. *Let* $R = \ker r$ *be a closed subspace of a normed linear space* V*, where* r *maps* V *onto* E*. If we assume* (2.23), *the operator* p*, mapping* E *into* V*, defined by*

$$p = J^{-1} r' J_E\,, \tag{2.25}$$

satisfies the properties (2.20), (2.21), *and* (2.22) *of Corollary* 2.1.

Remark 2.2. In the same way as in Ref. [1], we can extend the last corollary to the case where V is equipped with a seminorm $p(v)$, instead of a norm $\| \; \|$.

2.4. *Characterization of the stabilization projector*

Let us assume now that a closed subspace P of V is the range of an isomorphism p from a Banach space E into V.

We shall compute the stabilization projector $s = s_P{}^{\alpha}$ in terms of p and of the duality mapping $J = J_V{}^{\alpha}$ from V onto V'.

THEOREM 2.3. *If* V *is an R.S. space, the stabilization projector* s *onto* $P = pE$ *is given by*

$$s = pr, \tag{2.26}$$

where $r = r^{\alpha}$ *is the H operator from* V *onto* E *with* $\ker r = P^{\oplus}$, *defined by*

$$r = (p'Jp)^{-1} p'J. \tag{2.27}$$

First of all, $p'Jp$ is equal to the duality mapping $J_E = J_E{}^{\alpha}$ of E when it is supplied with the norm $\| e \| = \| pe \|$ (Lemma 1.2). Therefore $(p'Jp)_E^{-1} = J_{\alpha'}^{E'}$, and thus $rp = 1$, and $s = pr$ is an H projector onto P.

Since $P^{\perp} = \ker p'$, we deduce from Theorem 2.2 that

$$1 - t^{\perp} = Jp(p'Jp)^{-1} p',$$

and thus

$$s = pr = J^{-1}(1 - t^{\perp})J.$$

Theorem 2.1 implies that s is the stabilization projector onto P.

COROLLARY 2.3. *Among the left-inverse H operators of p, the operator r, defined by* (2.27), *is the one which achieves the minimal norm* (*equal to one*).

Indeed, $\| ru \|_E = \| pru \|_V = \| su \|_V \leqslant \| u \|$, by Lemma 2.3. Therefore, $\| r \| = 1$, and any left inverse of p has a norm greater than one. Let us notice the following formula.

COROLLARY 2.4. *The operator r, defined by* (2.27), *is related to p by*

$$\cos_V(u, pe) = \frac{\| ru \|^{\alpha-1}}{\| u \|^{\alpha-1}} \cos(ru, e); \qquad u \in V, \quad e \in E. \tag{2.28}$$

3. CHARACTERIZATION OF THE ERROR AND STABILITY FUNCTIONS

3.1. *Stability functions and error functions of a subspace P*

Let us consider two Banach spaces U and V such that

$$U \text{ is contained in } V \text{ with a stronger topology}; \quad U \text{ is dense in } V. \tag{3.1}$$

Let P be a closed subspace of U and V. We associate with P the following functionals:

(i) *the error function*

$$e_U{}^V(P) = \sup_{u \in U} \inf_{v \in P} \frac{\| u - v \|_V}{\| u \|_U}; \tag{3.2}$$

(ii) *the stability function*

$$s_U{}^V(P) = \sup_{v \in P} \frac{\| v \|_U}{\| v \|_V}. \tag{3.3}$$

There is a dual relation between these two functionals.

THEOREM 3.1. *Let us Assume* (3.1) *and let* $P^\perp$ *be the annihilator of P. Then*

$$\text{(i)} \quad e_U{}^V(P) = s_{V'}^{U'}(P^\perp); \qquad \text{(ii)} \quad s_U{}^V(P) = e_{V'}^{U'}(P^\perp). \tag{3.4}$$

Indeed, let

(i) j be the canonical injection from U into V;

(ii) φ be the canonical surjection from V onto V/P. (3.5)

Then $e_U{}^V(P)$ is the norm, in $\mathscr{L}(U, V/P)$, of the operator $\varphi \cdot j$. By transposition, $e_U{}^V(P)$ is the norm of $j'\varphi'$ in $\mathscr{L}((V/P)', U')$.

But $(V/P)'$ is isometric to $P^\perp$, and so φ' can be identified with the canonical injection from $P^\perp$ into V'. Since U is dense in V, we can identify V' with a subspace of U', and j' becomes the canonical injection from U' into V'. Therefore

$$e_U{}^V(P) = \| j'\varphi' \|_{\mathscr{L}(P^\perp, U')} = \sup_{f \in P^\perp} \frac{\| f \|_{U'}}{\| f \|_{V'}} = s_{U'}^{V'}(P^\perp).$$

One can prove (3.4(ii)) in the same way.

3.2. *Computation of the norm of an operator*

Let X and Y be R.S. spaces, Z a Banach space, and

$$\begin{aligned} &\text{(i)} \quad A \text{ a linear operator from } Z \text{ into } X; \\ &\text{(ii)} \quad B \text{ an isomorphism from } Z \text{ into } Y. \end{aligned} \tag{3.6}$$

Let us set

$$d(A, B) = \sup_{u \in Z} \frac{\| Au \|_X}{\| Bu \|_Y}. \tag{3.7}$$

By Lemma 1.2, the operator $B'J_Y{}^\alpha B$, mapping Z into Z', is invertible, since it is actually the duality mapping of Z, equipped with the norm $\| Bu \|_Y$.

We shall need the following:

THEOREM 3.2. *Assume that the* sup *in* (3.7) *is achieved at a point* u_0 *satisfying* $\| Bu_0 \|_Y = 1$. *Then* $d(A, B)$ *is the* α'-th *root of the largest eigenvalue of the operator* $(B'J_Y{}^\alpha B)^{-1} (A'J_X{}^\alpha A)$,

$$(B'J_Y{}^\alpha B)^{-1} (A'J_X{}^\alpha A)\, u_0 = d(A, B)^{\alpha'} u_0 ; \qquad \| Bu_0 \|_Y = 1. \tag{3.8}$$

Consider the functional

$$\rho(v) = \frac{\| Av \|_X^\alpha}{\| Bv \|_Y^\alpha}. \tag{3.9}$$

Since X and Y are smooth, this functional has a Gâteaux derivative Lu (for $u \neq 0$) satisfying

$$Lu = \frac{\| Bu \|_Y^\alpha A'J_X{}^\alpha Au - \| Au \|_X^\alpha B'J_Y{}^\alpha Bu}{\| Bu \|_Y^{2\alpha}}.$$

By hypothesis, there exists u_0 such that

$$\| Bu_0 \|_Y = 1; \qquad d(A, B)^\alpha = \rho(u_0) \geqslant \rho(v) \quad \text{for any} \quad v \neq 0.$$

Therefore, for any $v \neq 0$, $(Lu_0, v) = 0$ and thus,

$$Lu_0 = A'J_X^{\alpha}Au_0 - d(A, B)^{\alpha} B'J_Y^{\alpha}Bu_0 = 0.$$

Hence $d(A, B)^{\alpha'}$ is an eigenvalue of the operator $(B'J_Y^{\alpha}B)^{-1} A'J_X^{\alpha}A$. It is the largest one, since if $d^{\alpha'}$ is any positive eigenvalue, we deduce from

$$A'J_X^{\alpha}Au = d^{\alpha}B'J_Y^{\alpha}Bu$$

the equality

$$\| Au \|_X^{\alpha} = d^{\alpha} \| Bu \|_Y^{\alpha} .$$

Remark 3.1. We shall apply, in a forthcoming paper, this kind of result to the problem of existence of eigenvalues of nonlinear operators.

3.3. *Characterization of the stability and error functions*

Let U and V be R.S. spaces such 'hat

$$U \subset V, \text{ he injection being compact and dense,} \tag{3.10}$$

and let $K = K_U^{\alpha}$ and $J = J_V^{\alpha}$ be the duality mappings of U and V, respectively.

Let P be a closed subspace of V.

Let $t = t_P$ be the best-approximation projector onto P (in V), $P^{\oplus} = \ker t$, and $s = s_P^{\alpha}$ the stabilization projector onto P (in V).

THEOREM 3.3. *Let us assume* (3.10). *Then* $e_U^V(P)$ *is the* α'-th *root of the largest eigenvalue of the operator* $(1 - t) K^{-1}J(1 - t)$, *mapping* $P^{\oplus}$ *into itself,*

$$(1 - t) K^{-1}J(1 - t)u = e_U^V(P)^{\alpha'}u; \qquad u \in P^{\oplus}. \tag{3.11}$$

Furthermore, if we assume

$$P \text{ is a finite-dimensional subspace contained in } U \text{ such that } KP \subset V', \tag{3.12}$$

then $s_U^V(P)$ *is the* α'-th *root of the largest eigenvalue of the operator* $sJ^{-1}Ks$, *mapping* P *into itself,*

$$sJ^{-1}Ksv = s_U^V(P)^{\alpha'}v; \qquad v \in P. \tag{3.13}$$

Let us write $P = \ker r$, where r is a continuous linear operator from V onto a Banach space E (for instance, we can take $E = V/P$ and r the canonical surjection from V onto V/P). Then $P^{\perp} = r'E'$ and, by Theorem 3.1,

$$e_U^V(P)^{\alpha'} = s_{U'}^{V'}(P)^{\alpha'} = \sup_{e \in E'} \frac{\| r'e \|_{U'}^{\alpha'}}{\| r'e \|_{V'}^{\alpha'}}; \qquad \frac{1}{\alpha} + \frac{1}{\alpha'} = 1.$$

By (3.10), the injection from V' into U' is compact and the sup is achieved at some point e. Therefore, by Theorem 3.2, $e_U{}^V(P)^\alpha$ is the largest eigenvalue of

$$(rJ^{-1}r')^{-1}\, rK^{-1}r'e = e_U{}^V(P)^\alpha e,$$

or, equivalently, of the operator

$$J^{-1}r'(rJ^{-1}r')^{-1}\, rK^{-1}JJ^{-1}r'e = e_U{}^V(P)^{\alpha'} \cdot J^{-1}r'e, \tag{3.14}$$

since r' is an isomorphism from E' onto $P^\perp$ and J is a bijection from V onto V'.

Therefore, $u = J^{-1}r'e$ belongs to $P^\oplus$ (by Lemma 2.2), and we can write (3.14) in the form

$$(1 - t)\, K^{-1}J(1 - t)u = e_U{}^V(P)^{\alpha'}u$$

by Theorem 2.2.

Let us now assume (3.12) and write $P = pE$, where p is an isomorphism from E onto P (we can choose, for instance, $E = P$, and p the canonical injection).

Since the dimension of P is finite, the supremum in

$$s_U{}^V(P)^\alpha = \sup_{e \in E} \frac{\| pe \|_U^\alpha}{\| pe \|_V^\alpha} \tag{3.15}$$

is achieved at some point e. Therefore, by Theorem 3.2, $s_U{}^V(P)^{\alpha'}$ is the largest eigenvalue of the operator

$$(p'Jp)^{-1}\, p'Kpe = s_U{}^V(P)^{\alpha'}e, \tag{3.16}$$

or, equivalently, of the operator

$$p(p'Jp)^{-1}\, p'JJ^{-1}Kpe = s_U{}^V(P)^{\alpha'}\, pe, \tag{3.17}$$

since p is an isomorphism from E onto P and since JJ^{-1} is the identity on KP, by (3.12). Therefore $u = pe$ belongs to P, $su = u$, and, by Theorem 2.3, we can write (3.17) in the form

$$sJ^{-1}Ksu = s_U{}^V(P)^{\alpha'}u; \qquad u \in P.$$

COROLLARY 3.1. *Let u be an eigenvector of $(1 - t)\, K^{-1}J(1 - t)$, associated with $e_U{}^V(P)^{\alpha'}$. Then u belongs also to $P^\oplus$.*

Indeed, if v belongs to P, we have

$$\frac{\| u + v - t(u+v)\|_V^\alpha}{\| u + v \|_U^\alpha} \leqslant \frac{\| u \|_V^\alpha}{\| u \|_U^\alpha} \leqslant \frac{\| u + (v - t(u+v))\|_V^\alpha}{\| u \|_U^\alpha}.$$

Therefore, $\| u \|_\alpha^U \leqslant \| u + v \|_\alpha^U$ for any v in U.

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